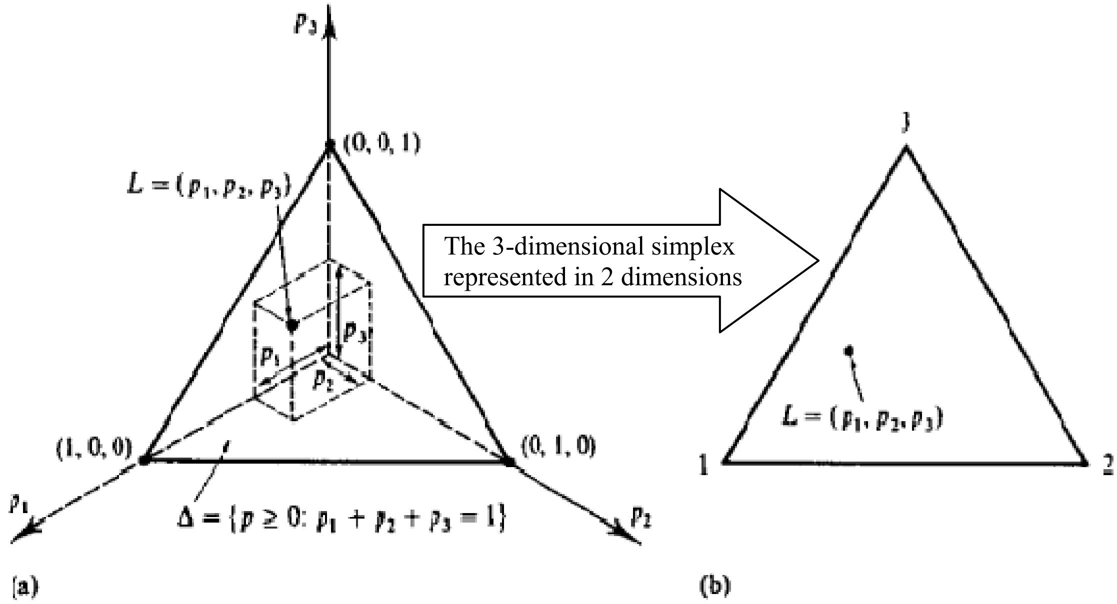
Uncertainty

- Let $C = \{c_1, \ldots, c_N\}$ be the ***set of all possible outcomes*** (or ***consequences***) of a risky alternative, where N is a finite number.

- A ***simple lottery*** L is a list $L = (p_1, \ldots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$. Here, p_n is the probability of the nth outcome occurring.
  - Two types of probabilities:
    - *Objective probability* is a probability that everyone can agree on.
    - *Subjective probability* depends on the individual perceiving the event.
  - A simple lottery can be represented geometrically as a point in the $(N-1)$ dimensional simplex, $\Delta = \{p \in \mathbb{R}_+^N : p_1 + \cdots + p_N = 1\}$.
    - A ***simplex*** is the set of all possible combinations of lotteries.
    - Each element in a simplex is also called a ***probability distribution***.



- A ***compound lottery*** $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ is a set of K simple lotteries L_k, each with a probability α_k of occurring, where $L_k = (p_1^k, \ldots, p_N^k)$ and $\alpha_k \in \mathbb{R}_+$ with $\sum_k \alpha_k = 1$.
  - $\alpha = (\alpha_1, \ldots, \alpha_K)$ is a simple lottery over another K simple lotteries.
  - The ***reduced lottery*** of a compound lottery is a simple lottery $L = (p_1, \ldots, p_N)$ where

$$p_n = \alpha_1 p_n^1 + \alpha_2 p_n^2 + \cdots + \alpha_K p_n^K = \sum_k \alpha_k p_n^k, \qquad \forall n \in \{1, \ldots, N\}.$$

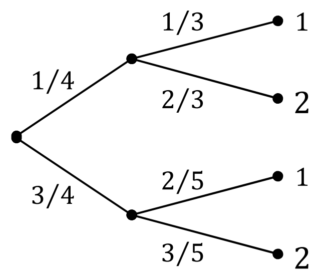
    - Note: $\sum_n p_n = \sum_k \alpha_k (\sum_n p_n^k) = \sum_k \alpha_k = 1$.
    - Note: A reduced lottery can result from more than one compound lottery (see below).

$N = K = 2$

$\alpha = \left(\frac{1}{4}, \frac{3}{4}\right)$

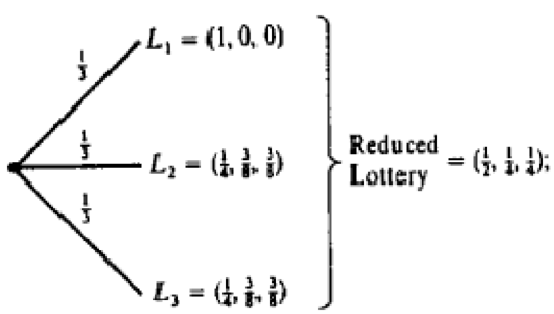
$L_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$

$L_2 = \left(\frac{2}{5}, \frac{3}{5}\right)$

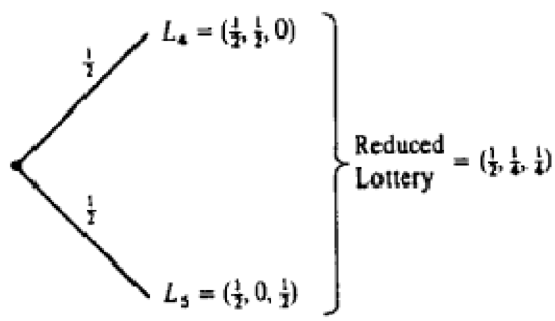


$$\begin{aligned} P(1,2) &= \left(\left(\frac{1}{4} \times \frac{1}{3} + \frac{3}{4} \times \frac{2}{5}\right), \left(\frac{1}{4} \times \frac{2}{3} + \frac{3}{4} \times \frac{3}{5}\right)\right) \\ &= \left(\left(\frac{1}{12} + \frac{6}{20}\right), \left(\frac{2}{12} + \frac{9}{20}\right)\right) \\ &= \left(\frac{23}{60}, \frac{37}{60}\right) \end{aligned}$$

$N = K = 3,\ \alpha = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$



$N = 3,\ K = 2,\ \alpha = \left(\frac{1}{2}, \frac{1}{2}\right)$

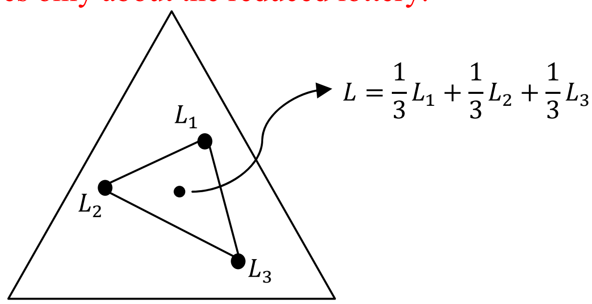


Expected Utility Theorem

❖ **Assumption 0: *Consequentialism***

➢ Indifference between a compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ and its associated reduced lottery $(p_1, \ldots, p_N)$.

- Individual cares only about the reduced lottery.



❖ **Assumption 1: *Rationality of the Preference Relation***

➢ Let $\mathcal{L}$ be the set of all simple lotteries over the set of outcomes C, and let $\succcurlyeq \subseteq \mathcal{L}^2$ be a binary preference relation on $\mathcal{L}$, i.e. for $L, L' \in \mathcal{L}$, $(L, L') \in \succcurlyeq \;\Leftrightarrow\; L \succcurlyeq L'$. The preference relation is ***rational*** if it is both complete and transitive:

- ***Completeness***: $\forall L, L' \in \mathcal{L} : (L \succcurlyeq L' \;\vee\; L' \succcurlyeq L)$
- ***Transitivity***: $\forall L, L', L'' : \big((L \succcurlyeq L') \wedge (L' \succcurlyeq L'')\big) \Rightarrow (L \succcurlyeq L'')$

❖ **Assumption 2: *Continuity of the Preference Relation***

➢ The preference relation is ***continuous*** if for any $L, L', L'' \in \mathcal{L}$, both of the following two sets are closed:

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succcurlyeq L''\}$$
$$\{\alpha \in [0,1] : L'' \succcurlyeq \alpha L + (1-\alpha)L'\}$$

➢ Continuity and Rationality together imply that $\succcurlyeq$ can be represented by a <u>continuous</u> utility function $V : \mathcal{L} \to \mathbb{R}$ such that

$$\forall L, L' \in \mathcal{L} : V(L) \geq V(L') \;\Leftrightarrow\; L \succcurlyeq L'$$

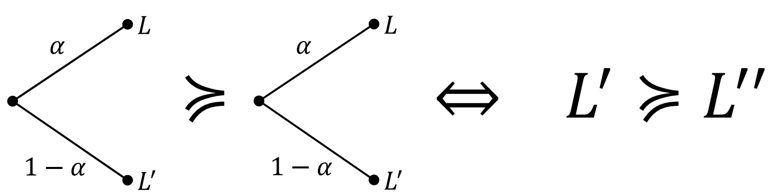
- However, for *expected utility*, we still need $u_1, \ldots, u_N$ such that $V(L) = \sum_{n=1}^{N} p_n u_n$

❖ **Assumption 3: *Independence*** (or ***Substitution***)

➢ For all L, L', L'', and for all $\alpha \in (0,1)$

$$\alpha L + (1-\alpha)L' \succcurlyeq \alpha L + (1-\alpha)L'' \;\Leftrightarrow\; L' \succcurlyeq L''$$

Graphically,



- ***Expected Utility Theorem***
  - A0, A1, A2, A3 imply that we can assign a set of numbers $(u_1, \ldots, u_N) \in \mathbb{R}^N$ to the set of outcomes $(c_1, \ldots, c_N)$ such that
$$\forall L, L' \in \mathcal{L} : L \succcurlyeq L' \quad \Leftrightarrow \quad \sum_{n=1}^{N} p_n u_n \geq \sum_{n=1}^{N} p_n' u_n$$
  - *Proof.* Suppose $1 \succcurlyeq 2 \succcurlyeq \cdots \succcurlyeq N$, and $1 \succ N$. Want to define $u(L)$ by the requirement that $L \sim \big(u(L), 0, \ldots, 0, 1 - u(L)\big)$. Define
$$S(L) = \{\alpha \in [0,1] : \alpha \mathbf{e}_1 + (1-\alpha)\mathbf{e}_N \succcurlyeq L\}$$
By A2, $S(L)$ is closed. We claim that $\alpha = 1 \in S(L)$:
$$\mathbf{e}_1 = (p_1 + \cdots + p_N)\mathbf{e}_1 \succcurlyeq (p_1 + p_3 + \cdots + p_N)\mathbf{e}_1 + p_2\mathbf{e}_2 \succcurlyeq \cdots \succcurlyeq p_1\mathbf{e}_1 + \cdots + p_N\mathbf{e}_N = L$$
    - This is true by virtue of the Independence Axiom.

Define
$$u(L) = \min\{\alpha \in S(L)\}$$
Need $u(L)\mathbf{e}_1 + \big(1-u(L)\big)\mathbf{e}_N \sim L$. We know that $u(L)\mathbf{e}_1 + \big(1-u(L)\big)\mathbf{e}_N \succcurlyeq L$. If $u(L)\mathbf{e}_1 + \big(1-u(L)\big)\mathbf{e}_N \succ L$, then $\exists \alpha < u(L) : \alpha \in S(L)$. But this is a contradiction.

Need $u(L) \geq u(L')$ if and only if $L \succcurlyeq L'$.
If $p > p'$ $\big(p = u\,\underbrace{(p\mathbf{e}_1 + (1-p)\mathbf{e}_N)}_{L}$ and $p' = u\,\underbrace{(p'\mathbf{e}_1 + (1-p')\mathbf{e}_N)}_{L'}\big)$,then by A3 and $\mathbf{e}_1 \succ \mathbf{e}_N$, we have
$$p\mathbf{e}_1 + (1-p)\mathbf{e}_N = p'\mathbf{e}_1 + (p-p')\mathbf{e}_1 + (1-p)\mathbf{e}_N \succ p'\mathbf{e}_1 + (1-p)\mathbf{e}_N$$

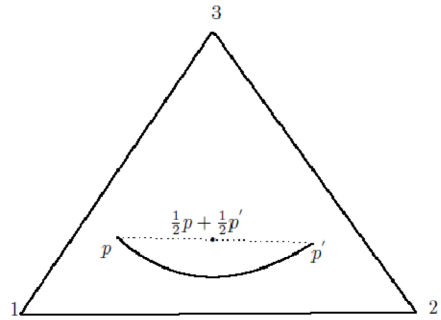
Need that $u(L) = \sum_{n=1}^{N} p_n u_n$.
Define $u_n = u(\mathbf{e}_n)$. We know
$$\begin{aligned} L &= p_1\mathbf{e}_1 + \cdots + p_N\mathbf{e}_N \\ &\sim p_1(u_1\mathbf{e}_1 + (1-u_1)\mathbf{e}_N) + p_2(u_2\mathbf{e}_1 + (1-u_2)\mathbf{e}_N) + \cdots \\ &\quad + p_N(u_N\mathbf{e}_1 + (1-u_N)\mathbf{e}_N) = \left(\sum_{n=1}^{N} p_n u_n\right)\mathbf{e}_1 + \left(1 - \sum_{n=1}^{N} p_n u_n\right)\mathbf{e}_N \end{aligned}$$

Q.E.D.

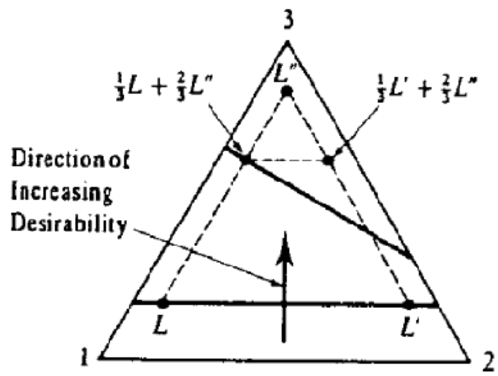
Expected Utility Theorem (cont'd)

- Note: the converse of the EUT also holds. That is, if $\succcurlyeq$ satisfy EUT, then A0, A1, A2, A3 hold.

- The preferences of lotteries must be linear, due to the Independence Axiom.



- The Independence Axiom also implies that indifference curves are parallel.



- *Proposition*. Suppose V and V' are both expected utility representations of $\succcurlyeq$. Then,
$$\exists \beta > 0, \exists \gamma \in \mathbb{R} : V' = \beta V + \gamma$$
where $u'_n = \beta u_n + \gamma$.

  - *Proof.* Suppose V' and u (from EUT) are EU.
$$V'(\mathbf{e}_n) = u_n V'(\mathbf{e}_1) + (1 - u_n)V'(\mathbf{e}_N)$$
$$u'_n = u_n u'_1 + (1 - u_n)u'_N = u_n(u'_1 - u'_N) + u'_N$$
  - Remark. The EU here is not only ordinal, but also cardinal!
    - However, any strictly monotone transformation of the EU still preserves the ordinality, but not the cardinality.

- The Allais Paradox
  - $C = (\$5\text{ million}, \$1\text{ million}, \$0)$
  - $L_1 = (0,1,0)$ and $L_2 = (0.10,0.89,0.01)$
  - $L'_1 = (0,0.11,0.89)$ and $L'_2 = (0.10,0,0.9)$
  - Naturally, $L_1 \succ L_2$, but it is likely that $L'_1 \prec L'_2$. This shows inconsistency of the expected utility theorem and the actual choice of individuals.

Lotteries over Money

- Suppose $V(p, w)$ is the Bernoulli utility function
  - Recall the indirect utility function:

$$V(p, w) = \max_x u(x), \qquad s.t. \quad p \cdot x = w$$

- $C = \mathbb{R}_+ = \{x : x \geq 0\}$
- $\mathcal{L} = \{F \mid F : \mathbb{R}_+ \to [0,1]\}$,
  where
  - F is non-decreasing;
  - right-continuous, i.e. $F(x) = \Pr\{\tilde{x} \leq x\}$; and
  - $\lim_{x \to \infty} F(x) \to 1$.
    - Note: There is no requirement for $F(0) = 0$.

  - F is called the ***cumulative distribution function*** (***cdf***).
  - If a probability density function, f, exists, then

$$F(x) = \int_0^x f(x)dx, \qquad f(x) = F'(x)$$

- Expected utility given F:

$$V(F) = \int u(x)dF(x)$$

  If F has a pdf,

$$V(F) = \int u(x)f(x)dx$$

Attitude towards Risk

- A gamble is ***fair*** if the expected total change of wealth in this gamble is zero.
- *Definition*. An individual is $\begin{cases} \text{risk-averse} \\ \text{risk-neutral} \\ \text{risk-preferring} \end{cases}$ if and only if $\int x dF(x) \begin{Bmatrix} \succcurlyeq \\ \sim \\ \preccurlyeq \end{Bmatrix} F$.

$$\int x dF(x) \begin{Bmatrix} \succcurlyeq \\ \sim \\ \preccurlyeq \end{Bmatrix} F \iff u\left(\int x dF(x)\right) \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} \int u(x)dF(x), \qquad \forall F \tag{1}$$

- *Proposition*. An individual is $\begin{cases} \text{risk-averse} \\ \text{risk-neutral} \\ \text{risk-preferring} \end{cases}$ if and only if u is $\begin{cases} \text{concave} \\ \text{linear (or affine)}. \\ \text{convex} \end{cases}$

  Proof. Suppose (1) holds, $F = (x, y; t, 1-t)$, $x, y \geq 0$, $t \in [0,1]$

$$u(tx + (1-t)y) \geq tu(x) + (1-t)u(x)$$

  This is the Jensen's Inequality.

- Application: Portfolio with a safe asset and a risky asset.
  - Safe asset: rate of return = 1
  - Risky asset: rate of return = $\tilde{z}$
  - Initial wealth: $W = \alpha + \beta$, where α is the amount invested in the risky asset, β is the amount invested in the safe asset.
$$(W - \alpha) + \alpha\tilde{z} = W + \alpha(\tilde{z} - 1)$$
  - Expected utility: $V(\alpha) = \int u\big(W + \alpha(\tilde{z} - 1)\big)dF(z) \quad \rightarrow \max_\alpha$
$$V'(\alpha) = \int u'\big(W + \alpha(z - 1)\big)(z - 1)dF(z)$$
$$V''(\alpha) = \int u''\big(W + \alpha(z - 1)\big)(z - 1)^2 dF(z) < 0$$
Assume that $u' > 0$ and $u'' < 0$.

⋮

  - Conclusion: No matter how risk-averse a person is, he will always invest some amount into the risky asset.
    - Because we assume that utility is continuous, when we look at the utility in a small neighborhood, the curve is approximately a straight line, which implies risk-neutrality.

- ***Coefficient of Absolute Risk Aversion***
$$r_A(x) = -\frac{u''(x)}{u'(x)}$$
  - Absolute risk-aversion is decreasing: $r_A'(x) < 0$.
    - Utility that has constant absolute risk-aversion:
$$u = -\frac{A}{c}e^{-cx} + B$$
  - E.g. $u(x) = -e^{-\alpha x}$, with
$$u'(x) = \alpha e^{-\alpha x}$$
$$u''(x) = -\alpha^2 e^{-\alpha x}$$

- Continue to show that as wealth increases, the amount an individual invests in a risky asset will increase as well (regardless of whether or not he is risk-averse).
  - See Han's notes.
  - Note: decreasing absolute risk aversion is crucial in proving this result.

- *Definition.* ***Coefficient of Relative Risk Aversion***

$$r_R(x) = -x \cdot r_A = -x\frac{u''(x)}{u'(x)}$$

  - Functions with ***constant relative risk aversion*** (***CRRA***):

$$\begin{aligned}
-x\frac{u''(x)}{u'(x)} = \rho > 0 \;&\Rightarrow\; \frac{u''(x)}{u'(x)} = -\frac{\rho}{x} \\
&\Rightarrow\; \ln u'(x) = -\rho \ln x \\
&\Rightarrow\; u'(x) = x^{-\rho} \\
&\Rightarrow\; \begin{cases} u'(x) = x^{-1}, & \rho = 1 \\ u'(x) = x^{-\rho}, & \rho \neq 1 \end{cases} \\
&\Rightarrow\; \begin{cases} u(x) = \ln x\,, & \rho = 1 \\ u(x) = \dfrac{x^{1-\rho}}{1-\rho}, & \rho \neq 1 \end{cases}
\end{aligned}$$

- **Stochastic Dominance**
  - ***First-Order Stochastic Dominance***. (Consider the class of c.d.f.'s with $F(0) = 0$ and $F(X) = 1$.)
    - *Definition.* F first-order stochastically dominates G if and only if, for all non-decreasing $u(\cdot)$

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

    - *Proposition***.** F first-order stochastically dominates G if and only if

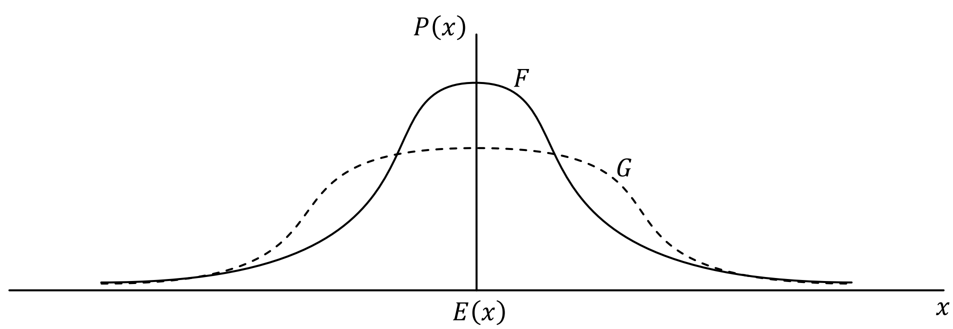
$$\forall x : F(x) \leq G(x).$$

Stochastic Dominance (cont'd)

- *Proposition***.** F first-order stochastically dominates G if and only if
$$\forall x : F(x) \leq G(x).$$
Proof. In both directions.
(⇒) Want to prove the contrapositive. Suppose $\exists \bar{x} : F(\bar{x}) > G(\bar{x})$. Let
$$u(x) = \begin{cases} 0 & x < \bar{x} \\ 1 & x \geq \bar{x} \end{cases}$$
Then,
$$\int udF = \int_{x \geq \bar{x}} dF = 1 - F(\bar{x})$$
$$\int udG = \int_{x \geq \bar{x}} dG = 1 - G(\bar{x}) > \int udF$$
(⇐) Suppose $F(x) \leq G(x)$ for all x. Take $u \in C^1$ with $u' \geq 0$. (Also, assume $F, G \in C^1$).
  - Assume $x \in [0, X]$, with $F(0) = G(0) = 0$ and $F(X) = G(X) = 1$.

  The trick is to use integration by parts.
$$\int_0^X udF = uF|_0^X - \int_0^X Fu'dx = u(X) - \int_0^X Fu'dx$$
$$\int_0^X udG = u(X) - \int_0^X Gu'dx$$
Then,
$$\int udF - \int udG = \int u'(G - F)dx \geq 0.$$

- ***Second-Order Stochastic Dominance***. Let $\int xdF = \int xdG$. Then F second-order stochastically dominates G if and only if, for all non-decreasing and concave u,
$$\int udF \geq \int udG$$
  - Focus on the mean and variance of the two probability distributions.
  - Mean-Preserving Spread:



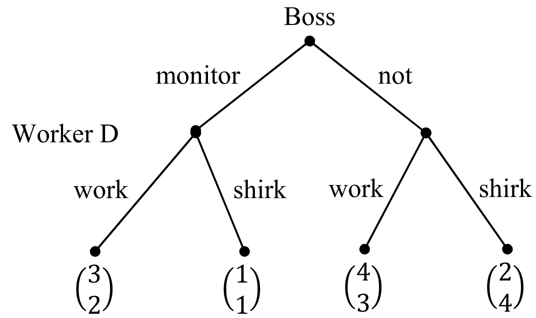
    - Let x be distributed according to F, and let $y = x + z$ where $z \in H_x$ such that $\int zdH_x(z) = 0$.
  - See Mas-Colell (p.198-199) for second representation of SOSD

- **Proposition.** If $\int xdF = \int xdG$, we have three equivalent conditions

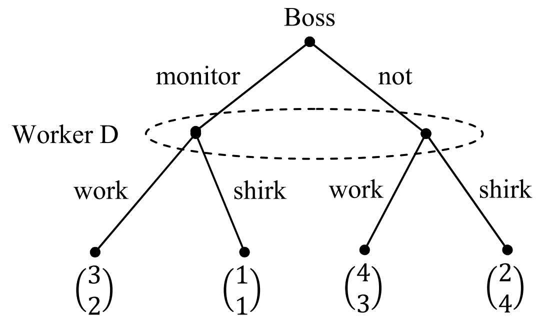
- F SOSD G
- G is a mean-preserving spread of F
- $\int_0^x G(t)dt \geq \int_0^x F(t)dt$ for all x

Game Thoery

❖ Extensive Form (Game Tree)



- This is a ***perfect information*** game.



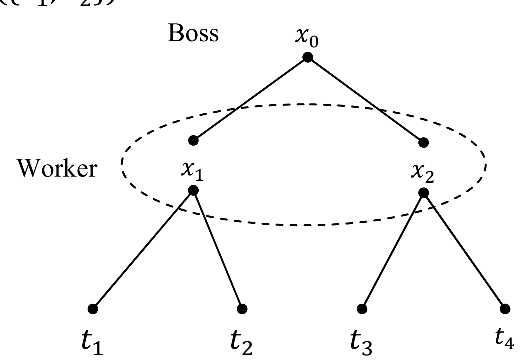
- This is a ***imperfect information*** game, where the dash-circle is an ***information set***.

❖ Basic specifications (to ensure that the game can be represented as a tree)

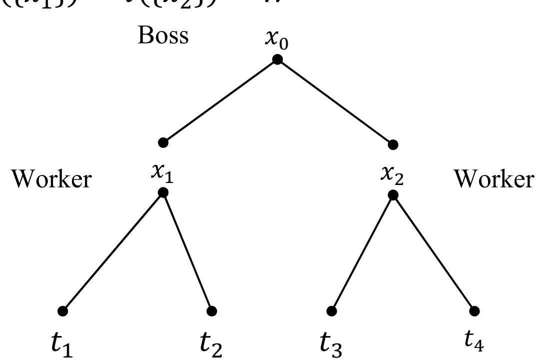
- Nodes (finite): $\mathcal{X}$
  - Terminal nodes: $T = \{x \in \mathcal{X} : s(x) = \emptyset\}$
  - Decision nodes: $D = \mathcal{X} \backslash T$
- Actions: $\mathcal{A}$
- Players: $\{1, \ldots, I\}$
- Predecessor function: $p : \mathcal{X} \to \mathcal{X} \cup \emptyset$, where $p^n(x) \neq x$ for all x and n, and $p(x) = \emptyset$ if and only if $x = x_0$.
  - This function allows us to move *backwards* on the game tree.
- Successor set: $s(x) = p^{-1}(x) = \{y : p(y) = x\}$
- Action last taken before x: $\alpha : \mathcal{X} \backslash \{x_0\} \to \mathcal{A}$
  - If $x, x' \in s(x)$ and $x \neq x'$, then $\alpha(x) \neq \alpha(x')$
- Set of choices: $C(x) = \{a \in \mathcal{A} : a = \alpha(x') \text{ for some } x' \in s(x)\}$
- Information set: A partition $\mathcal{H}$ of D, with $H : D \to \mathcal{H}$.
  - $H(x)$ is an information set, and $x \in H(x)$ for all x.
  - If $H(x) = H(x')$, then $C(x) = C(x')$.

Formal Setup of Extensive Form Game (cont'd)

- Basic specifications (to ensure that the game can be represented as a tree)
  - Nodes (finite): $\mathcal{X}$
    - Terminal nodes: $T = \{x \in \mathcal{X} : s(x) = \emptyset\}$
    - Decision nodes: $D = \mathcal{X} \backslash T$
  - Actions: $\mathcal{A}$
  - Players: $\{1, \dots, I\}$
  - Predecessor function: $p : \mathcal{X} \to \mathcal{X} \cup \emptyset$, where $p^n(x) \neq x$ for all x and n, and $p(x) = \emptyset$ if and only if $x = x_0$.
    - This function allows us to move *backwards* on the game tree.
  - Successor set: $s(x) = p^{-1}(x) = \{y : p(y) = x\}$
  - Action last taken before x: $\alpha : \mathcal{X} \backslash \{x_0\} \to \mathcal{A}$
    - If $x, x' \in s(x)$ and $x \neq x'$, then $\alpha(x) \neq \alpha(x')$
  - Set of choices: $c(x) = \{a \in \mathcal{A} : a = \alpha(x') \text{ for some } x' \in s(x)\}$
  - Information set: A partition $\mathcal{H}$ of D, with $H : D \to \mathcal{H}$.
    - $H(x)$ is an information set, and $x \in H(x)$ for all x.
    - If $H(x) = H(x')$, then $c(x) = c(x')$.
  - Decision makers: $\tau : \mathcal{H} \to \{0, 1, \dots, I\}$
    - $\tau(H)$ is the individual who moves at the information set H
    - For example, in the following tree,
      $\mathcal{H} = \{\{x_0\}, \{x_1, x_2\}\}$
      $\tau(\{x_0\}) = B$, and $\tau(\{x_1, x_2\}) = W$



Another example,
$\mathcal{H} = \{\{x_0\}, \{x_1\}, \{x_2\}\}$
$\tau(\{x_0\}) = B$, and $\tau(\{x_1\}) = \tau(\{x_2\}) = W$



  - $\mathcal{H}_i = \{H \in \mathcal{H} : \tau(H) = i\}$
- $\rho : \mathcal{H} \times \mathcal{A} \to [0,1]$
  - $\rho(H, a)$ is the probability of a at H
  - $\sum_{a \in c(H)} \rho(H, a) = 1$, and $\rho(H, a) = 0 \Leftarrow a \notin c(H)$
- Bernoulli utility: $u_i : T \to \mathbb{R}$, $i = 1, \dots, I$, and $T = \mathcal{X} \backslash D$

- Everything above define an extensive form game Γ_E.

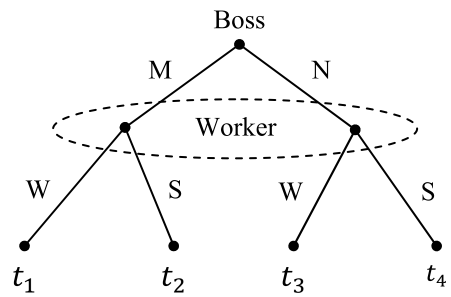
- *Definition*. A game Γ_E has ***perfect information*** if and only if $H(x) = \{x\}$ for all $x \in D$.
  - This is to say that all information sets are trivial.
  - We assume that players have ***Perfect Recall***, i.e. a player can remember every previous decision made by everyone up to the current one.

Normal / Strategic Form

- *Definition*. A ***strategy*** in an extensive form game Γ_E for i is

$$s_i : \mathcal{H}_i \to \mathcal{A}$$

$s_i(H) \in c(H)$ for all $H \in \mathcal{H}_i$. A strategy is a complete contingent plan.



B \ D	W	S
M	3,2	1,1
N	4,3	2,4

Assignment 1, Q1

For the case $N = 3$. Show that a utility representation for lexicographic preferences leads to a contradiction.

Let a lexicographic preference be as follows:

$$(p_1, p_2, 1 - p_1 - p_2) \succ (p_1', p_2', 1 - p_1' - p_2') \;\Leftrightarrow\; \begin{cases} p_1 > p_1', & \text{or} \\ p_1 = p_1' \;\wedge\; p_2 > p_2' \end{cases}$$

Suppose there exists a utility function $V : \mathcal{L} \to \mathbb{R}$ representing $\succcurlyeq$. Then,

$$V(p, 1 - p, 0) > V(p, 0, 1 - p), \qquad \forall p \in (0,1]$$

There must be a $q \in \mathbb{Q}$ such that

$$q \in \big(V(p, 0, 1 - p), V(p, 1 - p, 0)\big).$$

Notice that, given the characteristics of V, if there is a $p' > p$, then

$$V(p', 1 - p', 0) > V(p', 0, 1 - p') > V(p, 1 - p, 0) > V(p, 0, 1 - p).$$

Similarly, there exists a $q' \in \mathbb{Q}$ such that

$$q' \in \big(V(p', 0, 1 - p'), V(p', 1 - p', 0)\big).$$

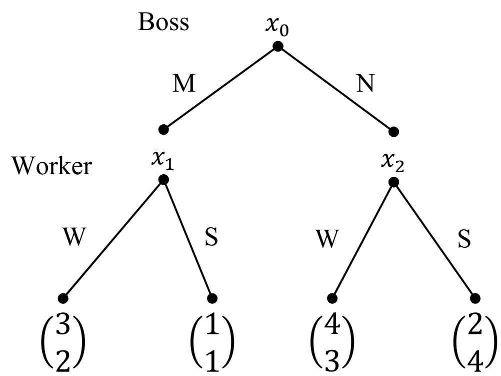
This means that we can have a mapping

$$\phi : [0,1) \to \mathbb{Q}$$

that is one-to-one. But this is impossible, as $\mathbb{Q}$ is countable and $[0,1)$ is uncountable.

Normal Form Game (cont'd)

❖ Boss v.s. Worker Game (with perfect information)



Boss \ Worker	WW	WS	SW	SS
M	3,2	3,2	1,1	1,1
N	4,3	2,4	4,3	2,4

- Given any finite extensive form game Γ_E, the normal (strategic) form Γ_N is composed of pure strategies
$$S_i = \{s_i : \mathcal{H}_i \to \mathcal{A}, \qquad s_i(H) \in c(H)\ \ \forall H \in \mathcal{H}_i\}$$
  - The number of pure strategies for a player is the product of the number of decisions at each information set raised to the power of the number of information sets.
  - The expected utility from the game is $u_i(s_1, \dots, s_I)$.
  - Choices of the s_i are simultaneous.
- Any extensive form game can be represented by a unique normal form game. However, there is no unique extensive form representation of a given normal form game.

Mixed Strategies

❖ *Definition.* Given S_i, the set of ***mixed strategies*** for player i is
$$\Delta(s_i) = \left\{ \sigma_i : S_i \to [0,1] \,\middle|\, \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}$$
Mixed strategies are independent across players.

❖ The payoff function (under mixed strategy)
$$\begin{aligned} U_i(\sigma_1, \dots, \sigma_I) &= \sum_{s \in S} \sigma_1(s_1) \cdots \sigma_I(s_I) u_i(s) \\ &= \sum_{s \in S} \left(\prod_{k \neq i} \sigma_k(s_k) \right) \cdot \left(\sigma_i(s_i) u_i(s_i, s_{-i}) \right) \\ &= \sum_{s \in S} \left(\prod_{k \neq i} \sigma_k(s_k) \right) \cdot \left(u_i(\sigma_i, s_{-i}) \right) \end{aligned}$$

where $s = (s_1, \dots, s_I)$ and $S = \prod_{i=1}^{I} s_i$.

- Given a mixed strategy profile $\sigma = (\sigma_1, \dots, \sigma_i, \dots, \sigma_I)$, the payoff function for player i is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{}(S$$

❖ *Definition.* A ***behavior strategy*** for i is

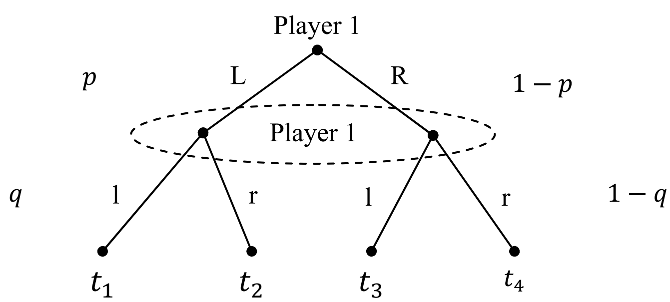
$$\lambda_i : \mathcal{H}_i \times \mathcal{A} \to [0,1]$$

where $\lambda_i(H, a)$ is the probability that i chooses a in $H \in \mathcal{H}_i$.

$$\sum_{a \in c(H)} \lambda_i(H, a) = 1, \qquad a \notin c(H) \;\Rightarrow\; \lambda_i(H, a) = 0$$

- **Kuhn's Theorem.**
  - Given a Γ_E, for any λ_i, there exists a σ_i which generates the same distribution over the terminal nodes T, regardless of the other players' types of strategies.
    - The set of mixed strategies is at least a greater set than the set of behavioral strategies.
  - Given any Γ_E with perfect recall, for any σ_i there exists a λ_i which generates the same distribution over the terminal nodes T, regardless of the other players' types of strategies.

- Example.



	l (with q)	r (with $(1-q)$)
L (with p)	pq	$(1-p)q$
R (with $(1-p)$)	$p(1-q)$	$(1-p)(1-q)$

This can be represented by a mixed strategy with the same probability distribution. However, a mixed strategy

$$\frac{1}{2}Ll + \frac{1}{2}Rr$$

cannot be represented by a behavioral strategy.

Simultaneous Move Games

- Prisoner's Dilemma

P1 \ P2	U	A
U	3, 3	1, 4
A	4, 1	2, 2

  - A strictly dominates U for both players.

- *Definition.* A strategy $s_i'' \in S_i$ is ***strictly dominated*** by $s_i' \in S_i$ if and only if

$$u_i(s_i', s_{-i}) > u_i(s_i'', s_{-i}), \qquad \forall s_{-i} \in S_{-i} = \prod_{j \neq i} S_j$$

  - Don't have to consider the other players' payoffs.

- ***Iterated Deletion of Strictly (or Weakly) Dominated Strategies***.

B \ D	W	S
M	3,2	1,1
N	4,3	2,4

  - D does not have strictly dominated strategies.
  - B has a strictly dominated strategy, M, so we can take it out.
    - This implies D *knows* that B has a dominated strategy.
  - In this game, the solution is unique.
  - For reference, see Jehle & Reny, p 270-273.

- *Definition.* A strategy $s_i' \in S_i$ is weakly dominated by $s_i'' \in S_i$ if and only if

$$u_i(s_i'', s_{-i}) \geq u_i(s_i', s_{-i}), \qquad \forall s_{-i} \in S_{-i}$$

where the inequality is strictly for some $s_{-i} \in S_{-i}$.
  - Example

R \ J	D	M
D	0, 0	0, 0
M	0, 0	100, 100

- ***Complete Information***: all players know the game, especially the other players' payoffs.
  - This assumption is needed to analysis a game using dominated strategies
- ***Rationality***: a dominated strategy will never be played
  - Every player is rational and he knows that the other players in the game are *rational*.
    - This is to say that rationality is common knowledge.
  - If all statements of the form

D knows that B knows that … B knows x

are true, then x is ***common knowledge***.

Iterated Deletion of (Strictly or Weakly) Dominated Strategies

- Stage 1: take out all strictly/weakly dominated strategies
- Stage 2: repeat stage 1 on the smaller game
- … keep repeating this procedure until it is not possible to do any more.
  - If reach a unique outcome, then this is the "solution".

- Two ways to delete dominated strategies
  - C.f. J&R pp. 270-273.
  - In MSG, each stage consists of deleting one strategy only

  - If a strategy to be deleted is *strictly* dominated, then it doesn't matter which way, J&R or MSG, we use to perform the iterated deletion.
    - If a strategy is strictly dominated, it is *always* strictly dominated!! Therefore, the order of deletion does not matter for strictly dominated strategies.

  - If a strategy is *weakly* dominated, then it's a little complicated. Example:

P1 \ P2	L	R
U	5, 1	4, 0
M	6, 0	3, 1
C	6, 4	4, 4

For P1, both U and M are weakly dominated by C. Suppose we take out U first,

P1 \ P2	L	R
M	6, 0	3, 1
C	6, 4	4, 4

P1 \ P2	R
M	3, 1
C	4, 4

P1 \ P2	R
C	4, 4

However, suppose we take out M first,

P1 \ P2	L	R
U	5, 1	4, 0
C	6, 4	4, 4

P1 \ P2	L
U	5, 1
C	6, 4

P2 \ P1	L
C	6, 4

Thus, different order of deletion of weakly dominated strategies may lead to different "solutions" of the game.

If we follow the J&R procedure, i.e. taking out *all* dominated strategies:

P2 \ P1	L	R
C	6, 4	4, 4

We get "solutions" from both orders of deletion in the MSG approach.

- Common Knowledge
  - "Common knowledge of x" is NOT equivalent to "everyone knows x"

Mixed strategies and Iterated Deletion of Dominated Strategies

- *Definition*. A mixed strategy $\sigma_i' \in \Delta(S_i)$ is strictly dominated if and only if

$$\exists \sigma_i'' \in \Delta(S_i), \forall \sigma_{-i} \in \prod_{j \neq i} \Delta(S_j) : u_i(\sigma_i'', \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i})$$

  - Example

P2 \ P1	L	R
U	10, .	0, .
M	4, .	4, .
D	0, .	10, .

M is not strictly dominated by U or D, but it is strictly dominated by $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ on (U, M ,D)

Nash Equilibrium

- Example

P2 \ P1	L	C	R
U	0, **4**	**4**, 0	5, 3
M	**4**, 0	0, **4**	5, 3
D	3, 5	3, 5	**6**, **6**

There are no dominated strategies for either player in this game.

- *Definition*. A strategy profile $s^* \in S$ is a ***Nash equilibrium***, if and only if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*), \qquad \forall s_i' \in S_i$$

  - If iterated deletion of strictly dominated strategies gives a "solution", it is a NE. Furthermore, this is the unique NE. (the procedure never takes out a NE)
  - If iterated deletion of weakly dominated strategies gives a solution, it is a NE. However, this NE may not be unique. (c.f. the Romeo and Juliet game from last class)
  - Problem of multiplicity of NE's. Example: Battle of sexes

P1 \ P2	V	P
V	**2**, **1**	0, 0
P	0, 0	**1**, **2**

There are two (pure strategy) NE's (and also one mixed strategy NE).

- Rationales for NE (c.f. MWG p.248)
  - Rational-expectations.
    If any theory makes a unique prediction in a game, it must be a NE.
  - Self-enforcing contract

  - Stable Social Convention

Mixed Strategies Nash Equilibrium

- *Definition.* The mixed strategy profile σ^* is a NE if and only if for all i
$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \qquad \forall \sigma_i' \in \Delta(S_i)$$
  - In a finite game, the existence of a mixed strategy NE is guaranteed.
  - Example

P1 \ P2	F	C	$\frac{1}{2}F + \frac{1}{2}C$
F	1, -1	-1, 1	
C	-1, 1	1, -1	
$\frac{1}{2}F + \frac{1}{2}C$			**0, 0**

- *Definition.* The ***support*** of σ_i is $S_i^+ = \{s_i \in S_i : \sigma_i(s_i) > 0\}$.
- **Proposition.** σ is a NE if and only if for all i
  - $u_i(s_i, \sigma_{-i}) = \sigma(s_i', \sigma_{-i}),\ \forall s_i, s_i' \in S_i^+$
  - $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}),\ \forall s_i \in S_i^+,\ \forall s_i' \notin S_i^+$

- Example (Battle of Sexes)

P1 \ P2	V	P	$qV + (1-q)P$
V	**2**, **1**	0, 0	
P	0, 0	**1**, **2**	
$pV + (1-p)P$			

  - If P1 randomizes, then his expected payoff should be the same by playing the two strategies:
$$2q + (1-q)(0) = 0q + (1-q)1$$
$$2q = 1 - q$$
$$q = \frac{1}{3}$$
  - Similarly, is P2 randomizes, her expected payoff should be equal by playing the two strategies:
$$1p + (1-p)(0) = (0)p + (1-p)(2)$$
$$p = 2(1-p)$$
$$p = \frac{2}{3}$$
  - A robust result: A finite game has an odd number of equilibria.

- **Theorem (Existence of Nash Equilibrium).** (c.f. J&R p. 270-280)
  - Every finite Γ_N has a Nash Equilibrium.

- **Brouwer's Fixed Point Theorem.**
  - Suppose $f : \Sigma \to \Sigma$ is continuous, where $\Sigma \subseteq \mathbb{R}^L$ is compact and convex. Then,
$$\exists \sigma^* \in \Sigma : f(\sigma^*) = \sigma^*.$$

<u>Existence of NE (cont'd)</u>

- Interpretation of MSNE: c.f. Harsanyi (1973, *International Journal of Game Theory*)

- *Proof of Existence of NE.*
  Notations:

$$\sigma_{ij} \equiv \sigma_i(s_j)$$
$$u_i(j, \sigma_{-i}) \equiv u_i(s_j, \sigma_{-i})$$
$$f_{ij} \equiv f_i(s_j)$$

Consider a continuous mapping $f : \Sigma \to \Sigma$ with compact and convex domain

$$f_{ij}(\sigma) = \frac{\sigma_{ij} + \max\{0,\ u_i(j, \sigma_{-i}) - u_i(\sigma)\}}{1 + \sum_j \max\{0,\ u_i(j, \sigma_{-i}) - u_i(\sigma)\}}.$$

Then, the Brouwer's Fixed Point Theorem implies that

$$\exists \sigma : f(\sigma) = \sigma.$$

$$\sigma_{ij} \sum \max\{0, \quad u_i(j, \sigma_{-i}) - u_i(\sigma)\} = \max\{0, \quad u_i(j, \sigma_{-i}) - u_i(\sigma)\}$$

$$\underbrace{\left(\sum_j \sigma_{ij}\{u_i(j, \sigma_{-i}) - u_i(\sigma)\} \max\{0, \quad u_i(j, \sigma_{-i}) - u_i(\sigma)\}\right)}_{=0}$$

$$= \sum_j (\{u_i(j, \sigma_{-i}) - u_i(\sigma)\} \max\{0, \quad u_i(j, \sigma_{-i}) - u_i(\sigma)\})$$

The bracketed term equals zero because

$$\underbrace{\sum_j \sigma_i(s_j) u_i(s_j, \sigma_{-i})}_{=u_i(\sigma)} - \underbrace{u_i(\sigma) \sum_j \sigma_i(s_j)}_{=u_i(\sigma)(1)} = 0$$

Thus,

$$\sum_j (\{u_i(j, \sigma_{-i}) - u_i(\sigma)\} \max\{0, \quad u_i(j, \sigma_{-i}) - u_i(\sigma)\}) = 0.$$

Since every term inside the summation sign is non-negative, it must be true that each term is equal to zero. (somehow this is what we want to prove, check J&R)

Games with Incomplete Information

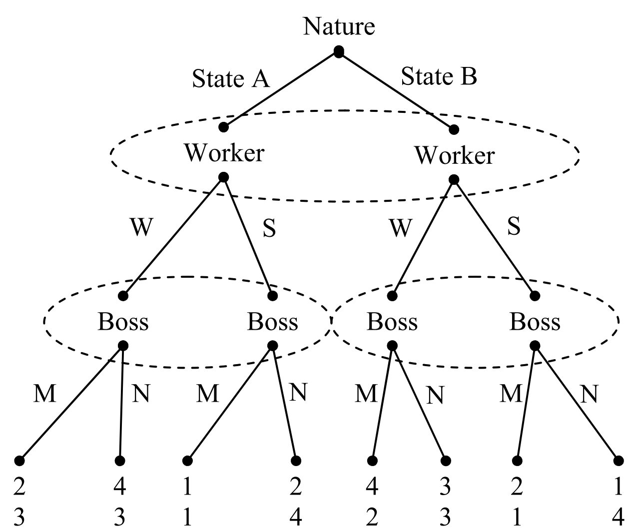
- Example: Boss v.s. Worker Game
  - In state-of-the-world A (with probability μ):

B \ D	W	S
M	3,2	1,1
N	4,3	**2,4**

  - In state-of-the-world B (with probability $1-\mu$):

B \ D	W	S
M	**4,2**	2,1
N	3,3	1,4

  - In extensive form



    - Boss should choose N in state A, and N in state B.
    - What should Worker choose?

$$W: 3\mu + (1-\mu)2 = 2 + \mu$$
$$S: 4\mu + (1-\mu) = 1 + 3\mu$$

The two functions equal when $\mu = 1/2$. Therefore, the Worker should choose to work if $\mu < 1/2$, and shirk if $\mu > 1/2$.

Midterm Answers

1. (i) False. Suppose there exists a continuous representation of preferences. Then, can take a monotonic transformation that is not continuous.

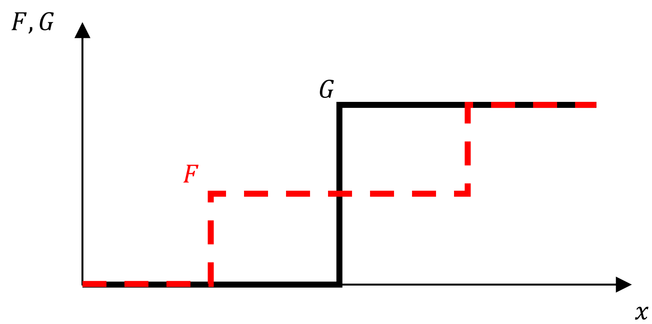
1. (ii) True.

$$V(\alpha) = \int u(w + \alpha(z-1)dF(z)$$
$$V''(\alpha) = \int u''\big(w + \alpha(z-1)\big)^2 (z-1)^2 dF(z) \geq 0 \quad \Leftarrow \quad u''(\cdot) \geq 0$$
$$V'(0) = u'(w)(Ez-1) > 0.$$

1. (iii) False.

$$F \succcurlyeq_{fsd} G \quad \Rightarrow \quad \int x dF(x) \geq \int x dG(x)$$

However, the reverse implication does not hold.



1. (iv) True. Terminal nodes are those nodes whose successor is empty.

1. (v) True.

2. (iv) $u'''(\cdot) > 0$ means that $u'(\cdot)$ is a convex function. Apply Jensen's inequality

$$\int u'(s+y)dF(y) > u'\left(\int (s+y)dF(y)\right) = u'(s), \qquad \forall s$$
$$u'(w-s^*) = \delta u'(s^*) < \int u'(s^*+y)dF(y) \quad \Rightarrow \quad V'(s^*) > 0 \quad \Rightarrow \quad \hat{s} > s^*$$

3. The sequence of S^i

$$q^{n+1} = \frac{1-q^n}{2} \quad \Rightarrow \quad q = \frac{1}{3}$$

Bayesian Games

- Idea: generalize games to scenarios where players have private information.
  - Example. I may know my private valuation of an object in an auction, but this information is not public (i.e. other players in the game do not have this information).
  - Even though the structure of Bayesian beliefs can be extremely complex, things finally resolve nicely

- Formal setup of Bayesian Game
  - I : set of players
  - Θ_i : set of possible types for player i
    - $\Theta = \times_{i\in I} \Theta_i$
    - $\theta = (\theta_1, \dots, \theta_I)$
  - $u_i(s_i, s_{-i}, \theta_i)$
  - $f : \Theta \to [0,1]$: this is a common prior for all players
    - For example, $f(\theta) = \prod_{i\in I} f_i(\theta_i)$ is an independent distribution.

  Thus, a Bayesian game is
$$\Gamma = \big\{I, \{S_i\}, \{\Theta_i\}, f, \{u_i\}\big\}.$$

- Example. Auction with two players
  - $I = \{1,2\}$
  - $\Theta_1 = \big\{\{0\},\{1\}\big\}$ and $\Theta_2 = \big\{\{0\},\{1\}\big\}$. Thus, $\Theta = \{(0,0),(0,1),(1,0),(1,1)\}$.
  - Common prior $P(\theta_1, \theta_2) = 1/4$.
  - $s_i = \{0, 1/N, 2/N, \dots, 1\}$
  - Payoffs:
$$u_i = \begin{cases} \theta_i - b_i & \text{if } b_i > b_j \\ 0 & \text{if } b_i < b_j \\ \dfrac{\theta_i - b_i}{2} & \text{if } b_i = b_j \end{cases}$$
  - Equilibrium.
    - *Ex-ante* (before i's type is revealed to i)
      - ***Pure strategies in Bayesian game*** is a map from Θ_i to S_i
        - $s_i(\cdot) : \Theta_i \to S_i$, thus $s_i(\theta_i) = s_i \in S_i$
      - Given a profile of pure strategies, $\big(s_1(\cdot), \dots, s_I(\cdot)\big)$, i's *ex-ante* payoff is
$$\begin{aligned}\tilde{u}_i\big(s_1(\cdot), \dots, s_I(\cdot)\big) &= E_\theta\left[u_i\Big(\big(s_1(\theta_1), \dots, s_I(\theta_I)\big), \theta_i\Big)\right] \\ &= \sum_{\theta\in\Theta} f(\theta) u_i\Big(\big(s_i(\theta_i), s_{-i}(\theta_{-i})\big), \theta_i\Big)\end{aligned}$$
    - ***Bayesian Nash Equilibrium*** is $\big(s_1(\cdot), \dots, s_I(\cdot)\big)$ such that
$$\tilde{u}_i\big(s_i(\cdot), s_{-i}(\cdot)\big) \geq \tilde{u}\big(s_i'(\cdot), s_{-i}(\cdot)\big), \qquad \forall s_i' \in S_i', \forall i \in I$$

- **Proposition.** $\big(s_1(\cdot), \dots, s_I(\cdot)\big)$ is a BNE if and only if
$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \theta_i)|\bar{\theta}_i] \geq E_{\theta_{-i}}[u_i(s_i'(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i)|\bar{\theta}_i]$$

Refinements (of NE)

- ***Trembling Hand Perfection*** (multiplicity of NE's).

R \ J	D	M
D	**0,0**	0,0
M	0,0	**100,100**

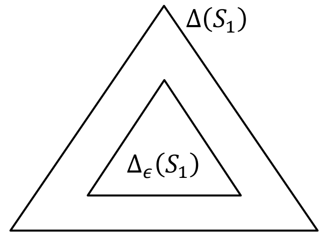
  - (D,D) is not a robust NE.
  - Suppose Juliet makes mistakes. For instance, she might mean to choose $(1,0)$, i.e. probability 1 on D and 0 on M. However, she ends up choosing $(\epsilon, 1-\epsilon)$ instead.

- Fix a normal form game $\Gamma_N = \{I, \{\Delta(s_i)\}, \{u_i\}\}$.
  Fix $\epsilon = \{\epsilon_1(\cdot), \dots, \epsilon_I(\cdot)\}$, where each $\epsilon_i : S_i \to \mathbb{R}_+$. E.g. $\epsilon_1(D) = \epsilon > 0$, and $\epsilon_1(M) = \epsilon' > 0$.
  $$\Delta_\epsilon(S_i) = \{\sigma_i : \sigma_i(s_i) \geq \epsilon_i(s_i), \qquad \forall s_i \in S_i\}$$
  Then, the game becomes
  $$\Gamma_\epsilon = \{I, \{\Delta_\epsilon(S_i)\}, \{u_i\}\}$$
  Want to consider Γ_{ϵ^k} with sequences $\epsilon^k = \left(\epsilon_1^k(\cdot), \dots, \epsilon_I^k(\cdot)\right)$

  - ***Trembling Hand Perfect Game Nash Equilibrium*** is Trembling Hand Perfect if and only if there exists a sequence of perturbed game
    $$\{\Gamma_{\epsilon^k}\} \to \Gamma_N$$
    with σ^k a NE of each Γ_{ϵ^k} and $\sigma^k \to \sigma$.

<u>Trembling Hand Perfection (cont'd)</u>

- Idea: want to look for an equilibrium that's robust w.r.t. small mistakes.
  - Force everybody to mix their strategies.



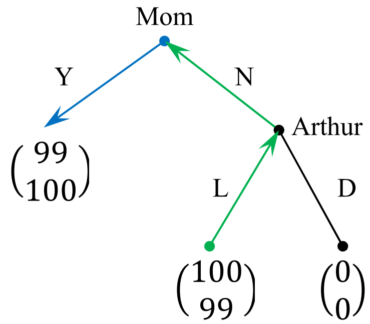
- **Proposition.** A NE σ of Γ_N is ***trembling hand perfect*** if and only if there exists a sequence of strategy profiles σ^k with totally mixed strategies and $\sigma^k \to \sigma$ as $k \to \infty$, such that σ_i is a best response to all σ_{-i}^k for all $\forall k$.
  - "Totally mixed" means $S_i^+ = S_i$ for every player i, i.e. every pure strategy of i is played with strictly positive probability: $\sigma_i^k(s_i) > 0$ for all i and all s_i.
  - σ_i must be robust to any small mistakes that i's opponents might make.
  - For proof, see Fudenberg and Tirole.

- **Proposition.** Suppose a NE σ is THP. Then, σ_i is not weakly dominated, for all i.
  - What about the converse?
    - Answer: If $I = 2$, any NE in (weakly) undominated strategies is THP. However, if $I \geq 3$, the converse is false. There exist NE in undominated strategies that are not THP.
    - This implies that THP can eliminate more than IDWDS.

- **Proposition.** Every finite game Γ_N has THPNE.
  - $\Gamma_{\epsilon^k} \to \Gamma_N$ such that $\sigma^k \to \sigma$.
    - The limit of a sequence of NE will also be a NE.
    - Every compact set has a convergent subsequence (i.e. σ^k).

<u>Dynamic Games (Extensive Form)</u>

❖ Example. Ice Cream between Arthur and Mom

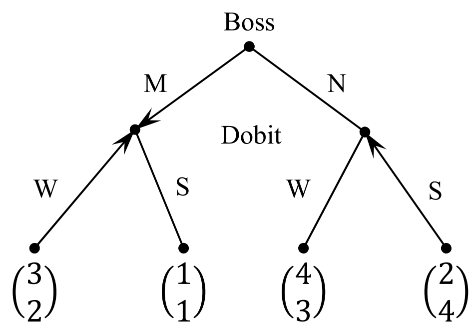


A / M	L	D
Y	99, 100	**99, 100***
N	**100, 99****	0, 0

- "Die" is an ***non-credible threat***. "Live" is weakly dominant!
- (*) is a NE
- (**) is ***Backward Induction NE*** (or Subgame Perfect Equilibrium)

❖ Example. Boss v.s. Dobit

D / B	WW	WS	SW	SS
M	3,2	**3,2****	1,1	1,1
N	4,3	2,4	4,3	**2,4***



- IDWDS will yield (M, WS).
- Backward induction yields (M, WS)

❖ ***Sequential Rationality***: each player optimizes at each information set (in the case of perfect information, each node is an information set).

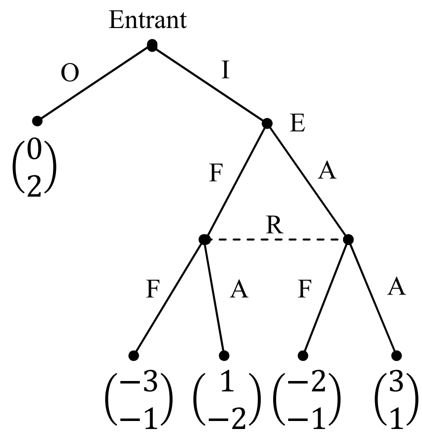
❖ ***Backward Induction*** (***BI***). Start at the nodes whose successors are only terminal nodes. Force an optimal choice. Take these choices as given, hence the game is reduced. Repeat the procedure. This must end by prescribing a complete contingent plan for all players.

- If there is no ties in a player's payoffs, then this procedure corresponds to IDWDS.

- **Proposition (Zermelo).** Every extensive form game Γ_E has a pure strategy Nash equilibrium that arises by BI.
  - This is not only a NE, but also a refinement (via BI) of NE.
  - If there are no ties, this is a unique BINE.

- ***Subgame Perfect Equilibrium*** (***SPE***)
  - Games with imperfect information

- Example.



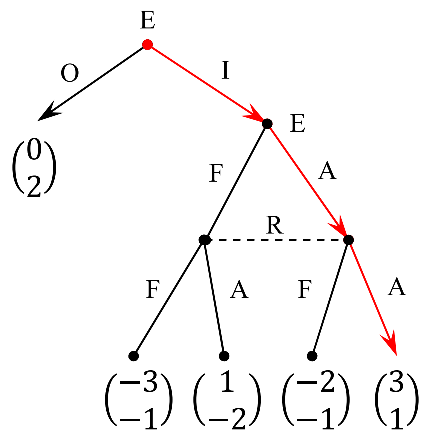
E \ R	F	A
F	-3,-1	1,-2
A	-2,-1	3,1**

<u>Trembling Hand Perfect NE (remark)</u>

- A trembling hand perfect NE may not necessarily survive the IDWDS.

<u>Subgame Perfect Equilibrium (cont'd)</u>

- Game from last time



E \ R	A	F
OA	2, 0	**2, 0***
OF	2, 0	**2, 0***
IA	**3, 1****	-2, -1
~~IF~~	~~1, -2~~	~~-3, -1~~

- Procedure is the same as backward induction.

- *Definition.* A ***subgame*** of Γ_E is a subset of Γ_E such that
  1) It starts with a singleton information set $H(x) = \{x\}$ and contains all and only all of the successors of x (and successors of its successors).
  2) If x' is in the subgame, so is any $x' \in H(x')$.

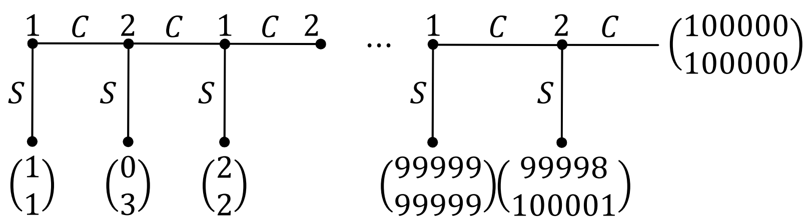
  - A subgame is a new Γ_E.
  - The original game Γ_E is a subgame of itself.
  - If Γ_E has perfect information, every x defines a subgame.

- *Definition.* $\sigma \in \Sigma$ is a ***subgame perfect equilibrium*** (or SPE) if and only if it generates a NE on every subgame.
  - $\{SPE\} \subseteq \{NE\}$
  - If Γ_E has perfect information, then $\{SPE\} = \{BINE\}$

- Connection between SPE and IDWDS.

- Both can be powerful, depending on the circumstances. For example, in the Romeo v.s. Juliet game, SPE doesn't take you very far in terms of getting rid of implausible NE's, but IDWDS will bring you to the plausible one. On the other hand, if there are ties in the payoffs (with perfect information), then SPE will be more useful.

- Centipede Game (Rosenthal)



  - Backward induction will show that S will be chosen at every stage.
  - In this case, BI is the same as IDWDS.

Dynamic Game (cont'd)

- Centipede game (from last time)
  - When is it rational for P1 to choose C? → When he thinks that P2 is also going to choose C. But P2 would be nuts if she chooses C!
  - A small number of nutty players would make everyone's payoff higher in the game.
    - Cf. Kreps, Milgrom, Roberts, and Wilson (1982) JET.

- Example in MWG p.265 8.F.2

1 \ 2	L	R
U	1,1,1	1,0,1
D	1,1,1	0,0,1

1 \ 2	L	R
U	1,1,0	0,0,0
D	0,1,0	1,0,0

  - For 2: L is best response to D and B_1
  - For 3: B_1 is best response to D and L
  - For 1: D is best response to R and B_2
  - To see that (D,L,B_1) is THP
    - 2 chooses $(1-\epsilon_2, \epsilon_2)$
    - 3 chooses $(1-\epsilon_3, \epsilon_3)$
    - 1's expected payoff from playing U:

$$
\begin{aligned}
&(1-\epsilon_2)(1-\epsilon_3)\underbrace{\pi_1(U,L,B_1)}_{=1}\\
&+(1-\epsilon_2)\epsilon_3\underbrace{\pi_1(U,L,B_2)}_{1}\\
&+\epsilon_2(1-\epsilon_3)\underbrace{\pi_1(U,R,B_1)}_{=1}\\
&+\epsilon_2\epsilon_3\underbrace{\pi_1(U,R,B_2)}_{=0}\\
&=1-\epsilon_2\epsilon_3
\end{aligned}
$$

    - 1's expected payoff from playing D:

$$
\begin{aligned}
&(1-\epsilon_2)(1-\epsilon_3)\underbrace{\pi_1(D,L,B_1)}_{=1}\\
&+(1-\epsilon_2)\epsilon_3\underbrace{\pi_1(D,L,B_2)}_{=0}\\
&+\epsilon_2(1-\epsilon_3)\underbrace{\pi_1(D,R,B_1)}_{=0}\\
&+\epsilon_2\epsilon_3\underbrace{\pi_1(D,R,B_2)}_{=1}\\
&=1-\epsilon_2-\epsilon_3+2\epsilon_2\epsilon_3
\end{aligned}
$$

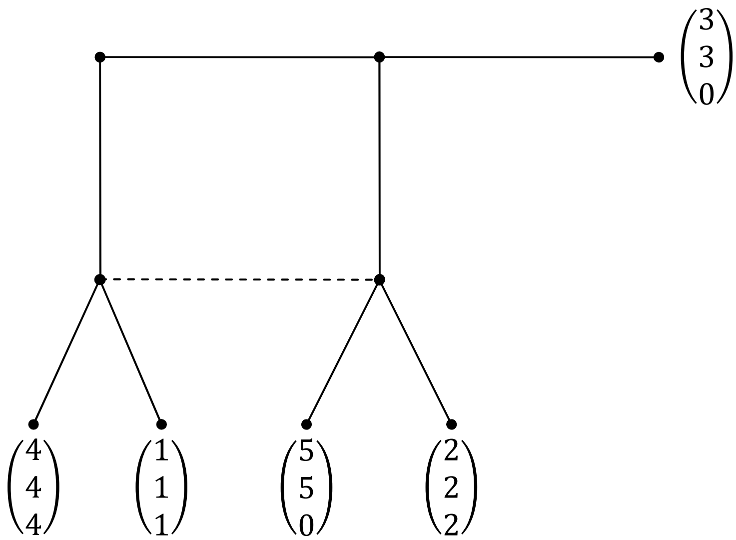
    - Comparing the two payoffs:

$$1-\epsilon_2\epsilon_3 > 1-\epsilon_2-\epsilon_3+2\epsilon_2\epsilon_3 \quad \Leftrightarrow \quad \epsilon_2+\epsilon_3 > 3\epsilon_2\epsilon_3$$

      If $\epsilon_2 < 1/3$, then $3\epsilon_2\epsilon_3 < \epsilon_3 < \epsilon_2+\epsilon_3$.
    - Therefore, as $\epsilon_2 \to 0$, P1 will be better off by choosing U, hence D is not the best response to all σ^k's.

Sequential Equilibrium

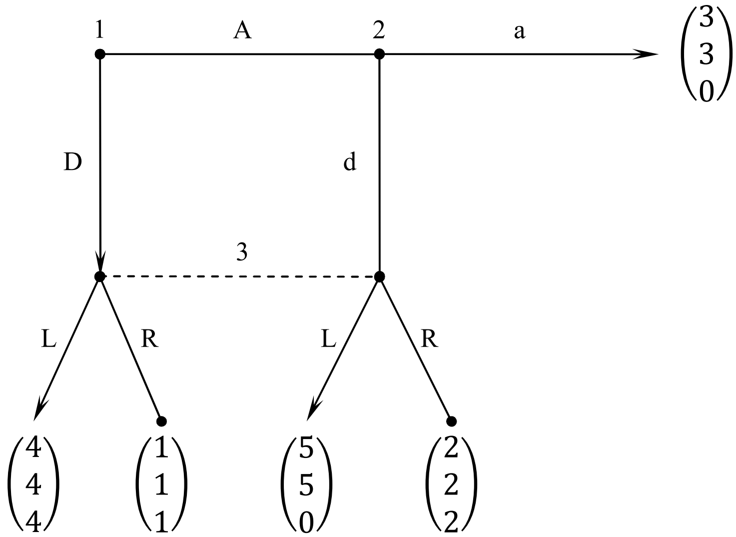
- Example in Kreps (1990, P.425-26):



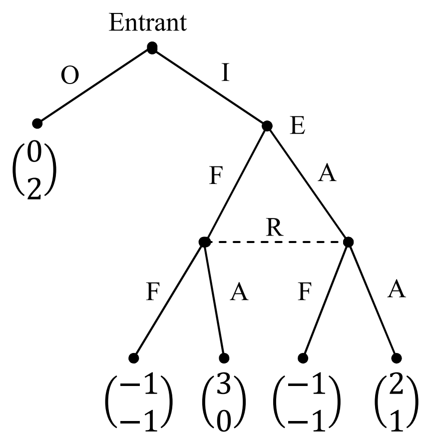
  - {NE} = {SPE}

Sequential Equilibrium (cont'd)

- The horse game from last time.

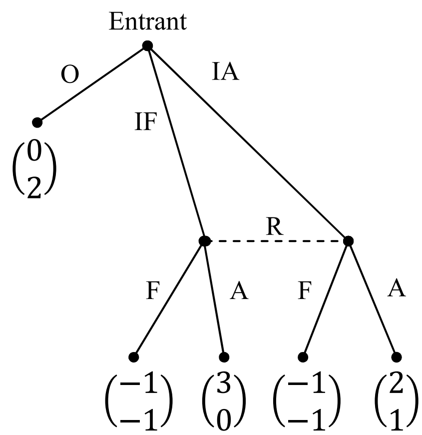


- Entrant game



E \ R	F	A
F	-1,-1	3,0**
A	-1,-1	2,1

  - (0,2) is not SPE
- However, in a modified entrant game, (0,2) is SPE:



- Therefore, subgame perfection depends on something artificial.

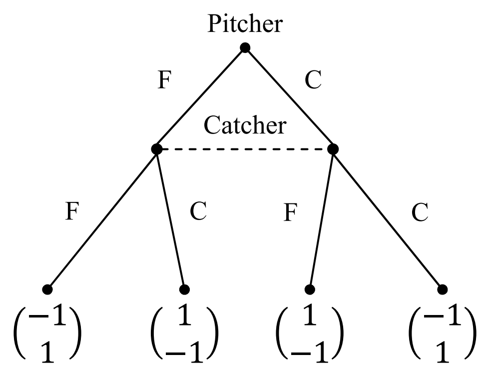
❖ *Definition.* A ***system of beliefs*** is

$$\mu : \underbrace{D}_{\substack{\text{set of}\\ \text{decision nodes}}} \to [0,1], \qquad s.t. \sum_{x \in H} \mu(x) = 1, \qquad \forall H \in \mathcal{H} \setminus \mathcal{H}_0$$

❖ *Definition.* A strategy profile σ is ***sequentially rational*** (***SR***) if and only if

$$E\left[u_{\tau(H)} \middle| H, \mu, \sigma_{\tau(H)}, \sigma_{-\tau(H)}\right] \geq E\left[u_{\tau(H)} \middle| H, \mu, \sigma'_{\tau(H)}, \sigma_{-\tau(H)}\right], \qquad \forall \sigma'_{\tau(H)} \in \Delta\left(S_{\tau(H)}\right), \forall H \in \mathcal{H}$$

❖ Example. Pitcher-Catcher game.



- If $\Pr\{H|\sigma\} > 0$, then

$$\mu(x) = \frac{\Pr\{x|\sigma\}}{\Pr\{H|\sigma\}}$$

This is the Bayes' rule.

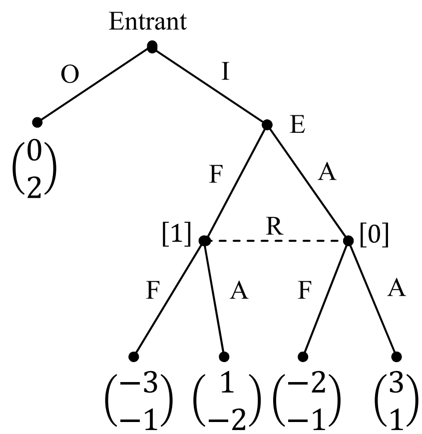
❖ *Definition.* (σ, μ) is a ***weak perfect Bayes equilibrium*** (***WPBE***) if and only if

(i) σ is sequentially rational given μ

(ii) μ satisfies Bayes' rule where applicable

- $\{WPBE\} \subseteq \{NE\}$

❖ Example.



- R has a non-credible threat F.
- (OA, F) is WPBE, but not SPE

- *Definition.* (σ, μ) is ***consistent*** if and only if

$$\exists\{\sigma^k\}_k \to \sigma : \mu^k \to \mu$$

where σ_k is a profile of completely mixed strategies and μ^k is derived by Bayes rule from σ^k.

- *Definition.* (σ, μ) is a ***sequential equilibrium***, ***SE***, if and only if
  - (i) (σ, μ) is consistent
  - (ii) σ is sequentially rational given μ

- **Theorem.** Every finite Γ_E has a SE.

- **Theorem.** If (σ, μ) is a SE, then σ is SPE.
  - $\{SE\} \subseteq \{SPE\}$

Asymmetric Information (Chpt. 13)

- Assumptions.
  - Many firms that are risk neutral, price-takers, with simple production technology that's constant return to scale. Price = 1.
  - Productivity of labor is $\theta \in [\bar{\theta}, \underline{\theta}]$, and is distributed according to CDF $F(\theta)$.
  - Opportunity cost of working for workers is $r(\theta)$, (with $r_\theta > 0$)

  - If θ is observable by firms, $w(\theta) = \theta$.
    - People who works are $\{\theta | \theta \geq r(\theta)\}$, this is Pareto efficient.

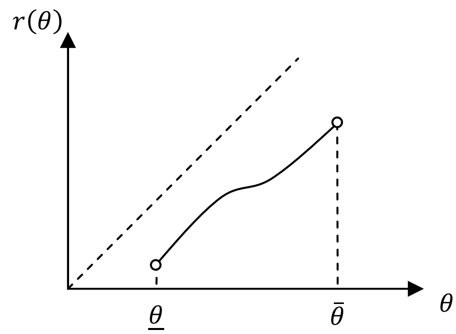
  - If θ is not observable by firms, then must have w independent of θ
    - Who works: $\{\theta | r(\theta) \leq w\}$
    - Wage should be $w = E\{\theta | r(\theta) \leq w\}$. Assume $r(\cdot)$ is continuous and strictly increasing. Then,
$$w = E\{\theta | \theta \leq r^{-1}(w)\} = g(w).$$

Asymmetric Information

- Recall from last time that

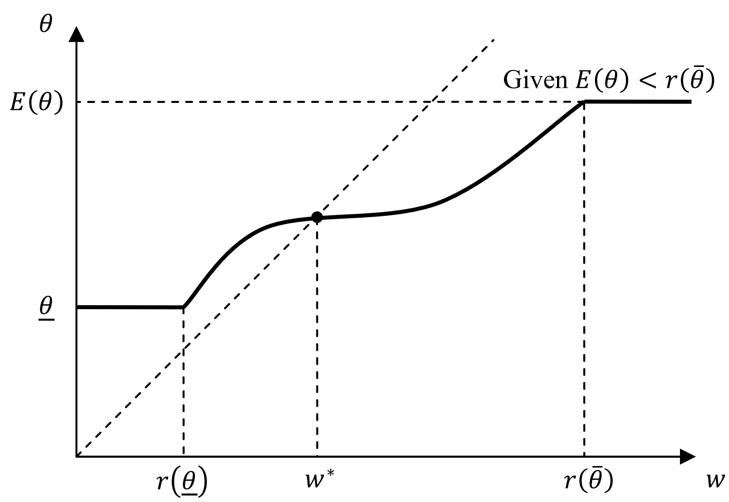
$$w = E\{\theta | r(\theta) \le w\} = g(w)$$

Assume that $r(\cdot)$ is continuous and increasing and $r(\theta) \le \theta$.

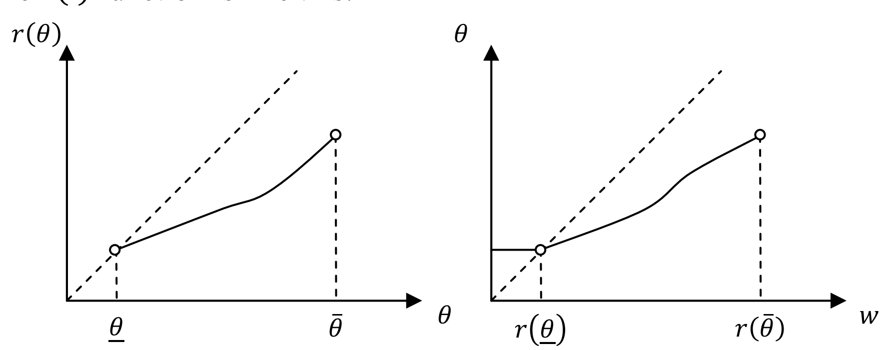


Also assume $F(\cdot)$ is continuous with density $f(\cdot)$.

$$g(w) = E\{\theta | \theta \le r^{-1}(w)\} = \frac{\int_{\underline{\theta}}^{r^{-1}(w)} \theta f(\theta) d\theta}{F\left(r^{-1}(w)\right)}$$

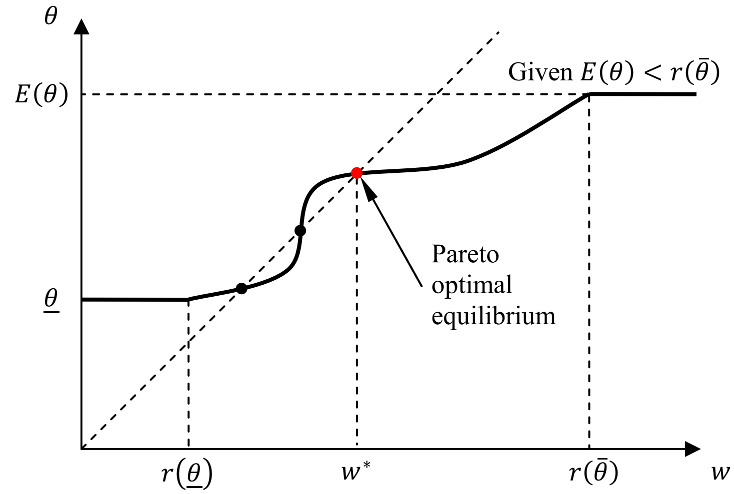


- This equilibrium is inefficient, because not everyone is working.
- Suppose the $r(\cdot)$ function is like this:



- The equilibrium may not be unique. However, in a robust sense, there will be a finite number of equilibrium, and the number will be odd.

- Consider a case where there are multiple equilibria



  - Note that the Pareto optimal equilibrium is efficient only in the second-best sense. Hence, the efficiency is a constrained-efficiency.

<u>Signaling (still in asymmetric information)</u>

- Education as a signal.
  - Two types:

$$\theta = \begin{cases} \theta_H & \text{with } \Pr\{\theta = \theta_H\} = \lambda \\ \theta_H > \theta_L > 0 & \text{with } \Pr\{\theta = \theta_L\} = 1 - \lambda \end{cases}$$

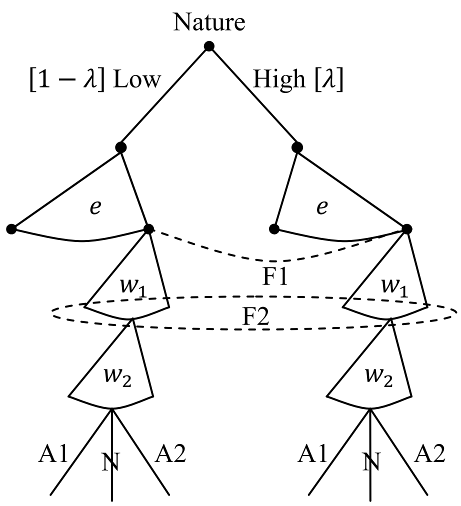
  - Cost of education $c(e, \theta)$ with

$$c_e > 0, \qquad c_{ee} > 0, \qquad c_\theta < 0, \qquad c_{e\theta} < 0$$

    - The condition $c_{e\theta} < 0$ is crucial.
    - It is cheaper for high productivity individual to acquire education.

Signaling Game (cont'd)

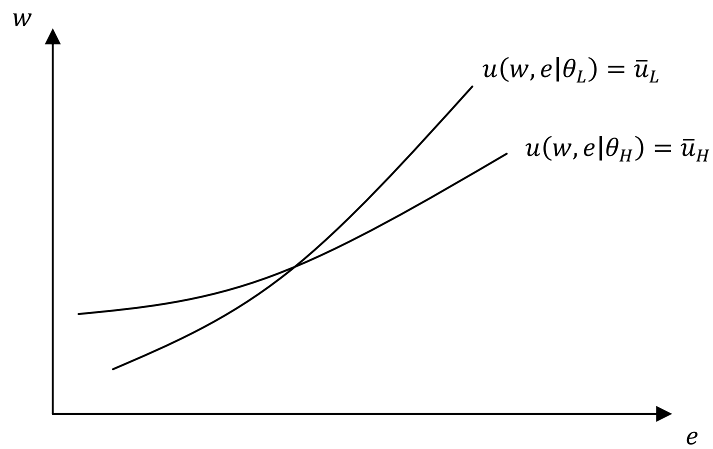
- Education as a signal of productivity.
  - Two types:
$$\theta = \begin{cases} \theta_H & \text{with } \Pr\{\theta = \theta_H\} = \lambda \\ \theta_H > \theta_L > 0 & \text{with } \Pr\{\theta = \theta_L\} = 1-\lambda \end{cases}$$
  - Cost of education $c(e,\theta)$ with
$$c_e > 0, \qquad c_{ee} > 0, \qquad c_\theta < 0, \qquad c_{e\theta} < 0, \qquad c(0,\theta) = 0$$
    - The condition $c_{e\theta} < 0$ is crucial.
    - It is cheaper for high productivity individual to acquire education.
  - Utility
$$u(w, e|\theta) = w - c(e,\theta)$$
  - Opportunity cost for participating in the labor market
$$r(\theta) = 0$$
    - This implies that everyone should work.
  - Extensive form game



  - Two firms playing a Bertrand game

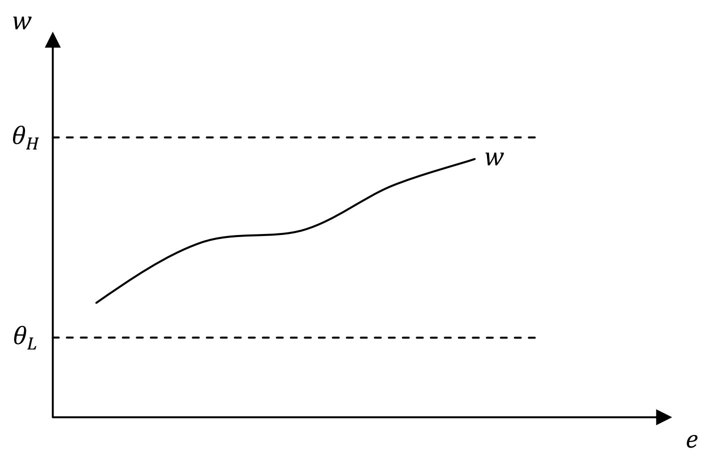
- ***Perfect Bayesian Equilibrium***
  - Workers' choice of $e, A1, A2, N$ is optimal given firms' choices.
  - Firms have common belief
$$\mu(e) = \Pr(\theta = \theta_H|e)$$
derived from Bayes' Rule whenever applicable.
  - Firms choices are a NE given each e and each associated $\mu(e)$
    - Suppose w is the only cost to the firm, and output is θ

- Firms' strategies
  - In a Bertrand game, firms set wage equal to expected output:
$$w(e) = \mu(e)\theta_H + \big(1-\mu(e)\big)\theta_L$$
- Workers' strategies
$$\frac{\partial w}{\partial e} = c_e(e,\theta)$$



- The indifference curves of the high and low types can cross at most once.

- The range of wages is

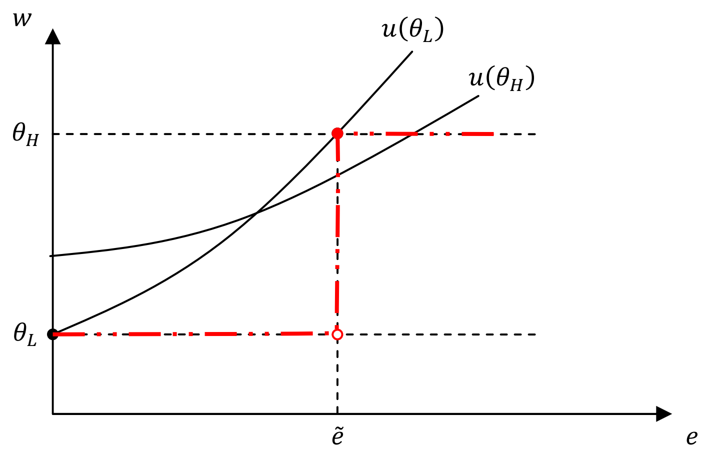


- *Definition.* ***Separating equilibrium*** is one such that $e(\theta_L) \neq e(\theta_H)$.

- **Lemma.** Given separating equilibrium, $w\big(e(\theta_H)\big) = \theta_H$ and $w\big(e(\theta_L)\big) = \theta_L$.
  - *Proof.* $\mu\big(e(\theta_H)\big) = 1$ and $\mu\big(e(\theta_L)\big) = 0$.

- **Lemma.** In the separating equilibrium, $e(\theta_L) = 0$.
  - This is the first best choice for the low type.

- Separating equilibrium (graphically)



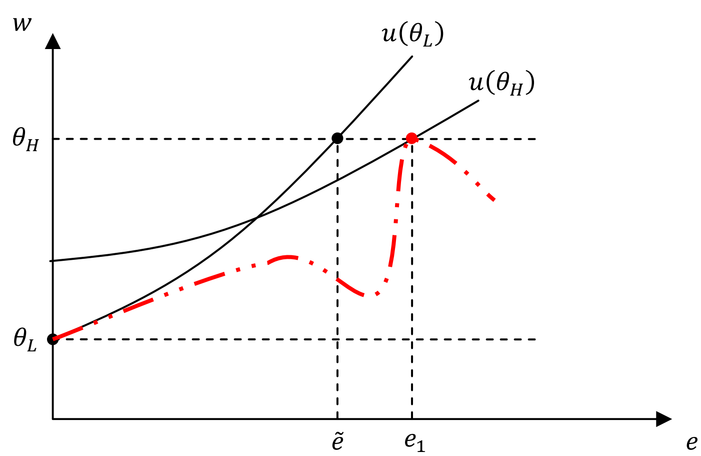
- This is a Perfect Bayesian Equilibrium.
- In this case, the firms' belief function is

$$\mu(e) = \begin{cases} 0 & e \in [0, \tilde{e}) \\ 1 & e \in [\tilde{e}, \infty) \end{cases}$$

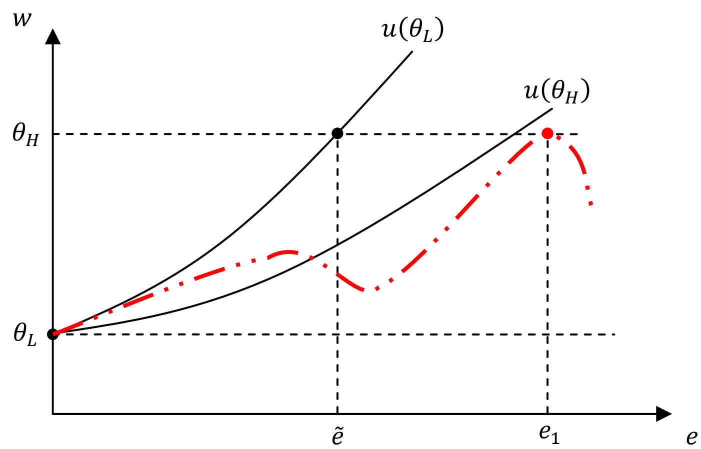
- The wage function is

$$w(e) = \begin{cases} \theta_L & e \in [0, \tilde{e}) \\ \theta_H & e \in [\tilde{e}, \infty) \end{cases}$$

- Other PBE can also be supported



- There is limit for how far right the firm can push e to

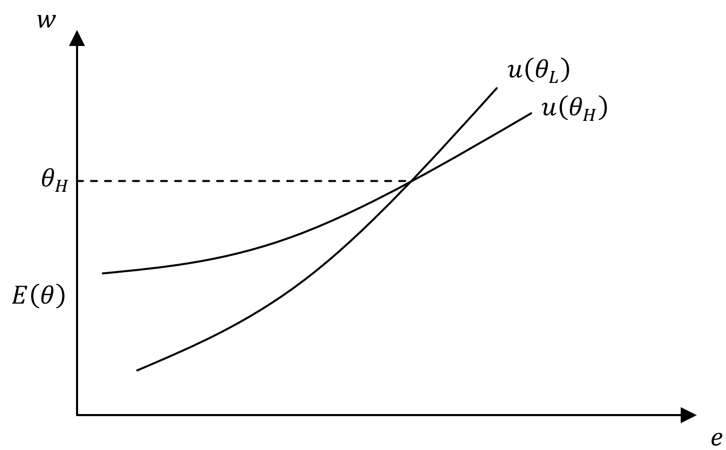


- In this case, even the high type would choose $e = 0$.

  - The equilibria as the firm increases e above $\tilde{e}$ are Pareto ranked: an equilibrium is Pareto superior if it is closer to $\tilde{e}$.

- What if we force $e = 0$? Then,

$$w = E(\theta) = \lambda\theta_H + (1-\lambda)\theta_L$$

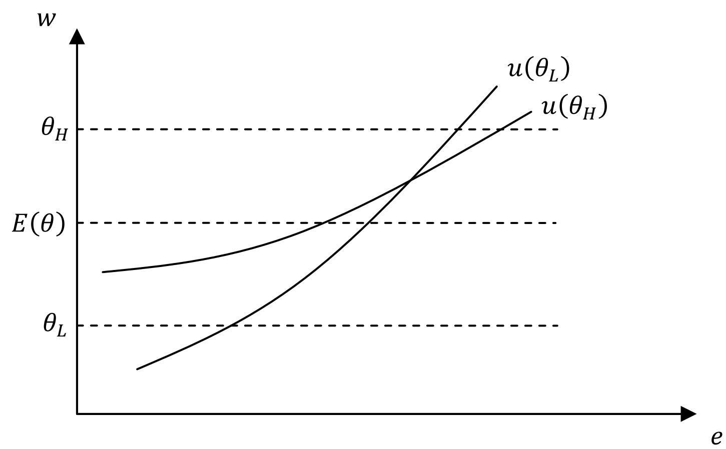


  - The low types are always worse off with signaling. The high types may or may not be better off (depending where $u(\theta_H)$ intersect the vertical axis).
    - As λ gets higher and higher, there are more high types, this works against the high types.

- *Definition.* ***Pooling equilibrium*** is one such that $e(\theta_H) = e(\theta_L)$.

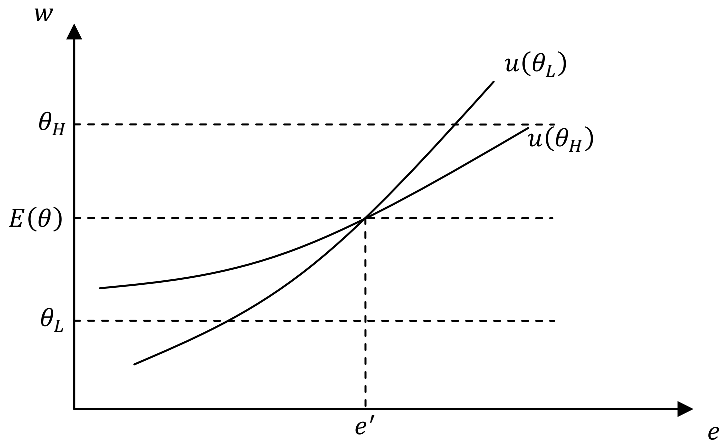
- $\mu\big(e(\theta_H)\big) = \lambda$ with $w\big(e(\theta_H)\big) = \lambda\theta_H + (1-\lambda)\theta_L = E(\theta)$

- Graphically,


w
$u(\theta_L)$
$u(\theta_H)$
θ_H
$E(\theta)$
θ_L
e

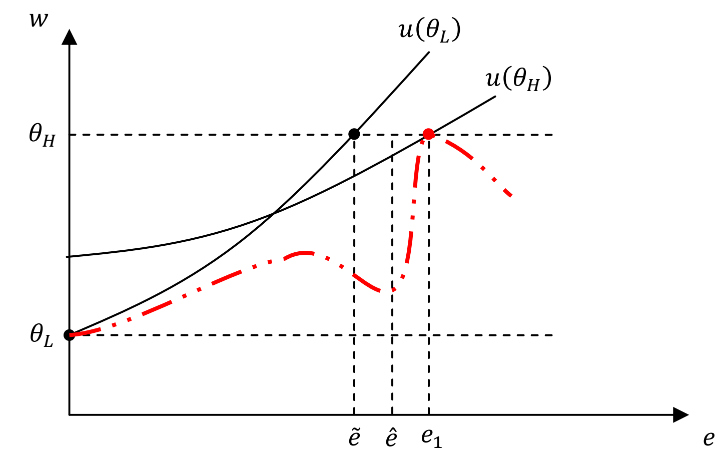
Signaling Game (cont'd)

❖ Pooling equilibrium



- Any $e \in [0, e']$ can be supported as a pooling equilibrium
- The Pareto efficient equilibrium is when $e = 0$.

❖ Equilibrium Refinement



If someone chooses $\hat{e}$, the firm believes with a positive probability that he is the low type. But no rational low type would ever choose $\hat{e}$!! Thus, $\mu(\hat{e}) \in (0,1)$, but no low type would choose $\hat{e}$.

- ***Intuitive criterion*** (Cho and Kreps) rules out all separating equilibria except $\tilde{e}$ and all the pooling equilibrium.
  - Check "stable equilibrium".

Introduction to Repeated Games

- Stage Game (Example 9.B.9)

1 \ 2	b_1	b_2	b_3
a_1	10, 10	2, 12	0, **13**
a_2	12, 2	**5**, **5**	0, 0
a_3	**13**, 0	0, 0	**1**, **1**

The game is played twice. Both player know what happens at stage 1.

For any SPE, either (a_2, b_2) or (a_3, b_3) must happen at stage 2, because they are NE's. What about at stage 1? NE must happen at stage 1 as well! So there are in total 4 SPE's.

Both can promise to play (a_1, b_1) at the first stage, and (a_2, b_2) at stage 2 if no deviation, and (a_3, b_3) if there is deviation.

See Benoit & Krishna (1985)

Review for Final

1. **(i)** False. Take a continuous function representation of a continuous preference, and make it jump somewhere.

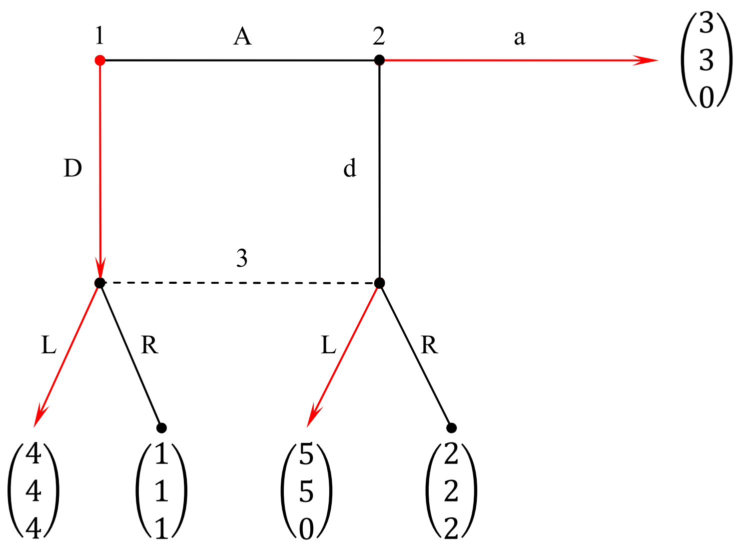
(ii) True. Consider

$$\frac{1}{2}\ln\left(\frac{w}{2}\right)+\frac{1}{2}\ln(w+p)>w \quad\Leftrightarrow\quad \frac{1}{2}\ln\left(\frac{w(w+p)}{2}\right)>\ln w \quad\Leftrightarrow\quad \frac{w(w+p)}{2}>w^2 \quad\Leftrightarrow\quad p>w$$

(iii) False, we need $r_A'(x)<0$ (DARA).

(iv) True.

(v) False. Consider the horse game.



2. **(a)** $v_i \sim U_{[0,1]}$ for $i=1,2$. $b_i:[0,1]\to[0,1]$. Ex-ante payoff:

$$\iint_{b_i(v_i)>b_j(v_j)}(v_i-b_i)dv_idv_j$$

(b) Interim payoff:

$$(v_i-b_i)\Pr\{b_j(v_j)<b_i\}$$

(c) Symmetric equilibrium:

$$\max_{\bar{b}}(v-\bar{b})\Pr\{b(\tilde{v})<\bar{b}\}$$

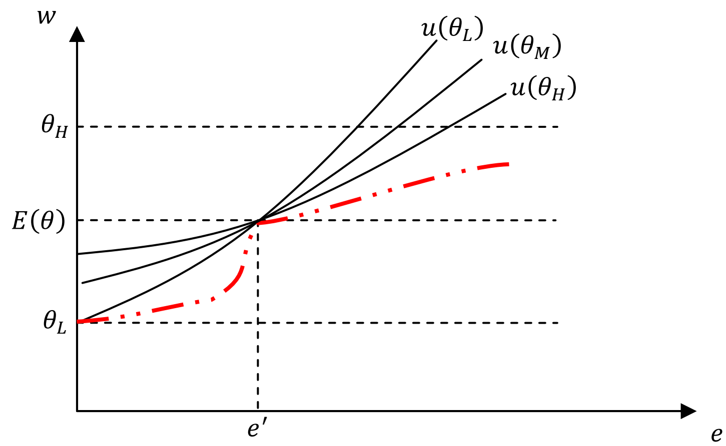
(d) Suppose $b(v)=\alpha+\beta v$. From (c),

$$(v-\bar{b})\Pr\left\{\tilde{v}<\frac{\bar{b}-\alpha}{\beta}\right\}=\frac{(v-\bar{b})(\bar{b}-\alpha)}{\beta}$$

Differentiate w.r.t. $\bar{b}$:

$$v - \bar{b} - \bar{b} + \alpha = 0 \;\Rightarrow\; \bar{b} = \frac{v + \alpha}{2} = \alpha + \beta v \;\Rightarrow\; \begin{cases} \alpha = 0 \\ \beta = 1/2 \end{cases}$$

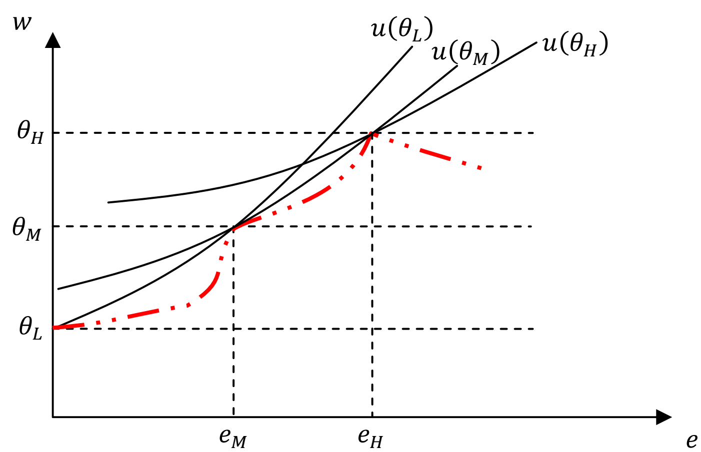
3. (a) $\theta_H > \theta_M > \theta_L > 0$.



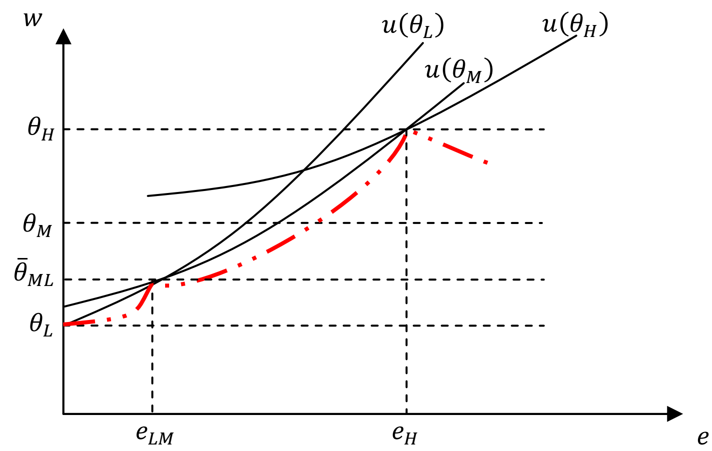
(b) Pareto preferred equilibrium is when $e' = 0$.

(c) $w^*\big(e^*(\theta_i)\big) = \theta_i$ for $i = L, M, H$ and $e^*(\theta_L) = 0$.

(d) Completely separating equilibrium:



(e) Partially separating equilibrium:


w
$u(\theta_L)$
$u(\theta_H)$
$u(\theta_M)$
θ_H
θ_M
$\bar{\theta}_{ML}$
θ_L
e_{LM}
e_H
e